A Conventional Foundation of Logic

Christian J. Feldbacher-Escamilla

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Introduction

Typical statements/schemata of the form

1. $7 + 5 = 12$
2. $\sim(\varphi \& \sim\varphi)$
3. 'All bachelors are unmarried.'

... are at the core of our notion of analyticity.

There are some approaches for "reducing" 1 to 2.

In this talk we are going to sketch a partial "reduction" from 2 to 3.

I.e.: We sketch a conventional foundation of logic.

Contents

The Unification of Analytic Truths

Frege's Logicism

Frege's *Grundlagen der Arithmetik* (1884):

> *"[The task of justifying is] that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one[.]" (cf. §3)*

Frege's logicism in a nutshell:

- Hume's Principle: $\#xFx = \#xGx$ iff $\exists f \forall x(Fx \to Gf(x)\ \&\ Gx \to Ff(x)\ \&\ \forall y \exists! z f(y) = z)$
- Definition of 0: $0 = \#xx \neq x$
- Definition of a successor function s: also by a 2nd-order formula (with $\#$ and logical symbols only)
- Definition of $\mathbb{N}$: 2nd-order formula

So, $\mathbb{N}$ (and math in general) can be reconstructed purely (2nd-order) logical.

Frege's Logicism: Just to mention

Hence: analytical = definitional + logical

Although Frege's approach failed . . .

"Nur in einem Punkte ist mir eine Schwierigkeit begegnet [. . .]"
(cf. Russell to Frege, June, 16, 1902):

- Frege's 2nd-order framework assumes class abstraction by co-exensionality—Basic Law V:
  $\{x : \varphi[x]\} = \{x : \psi[x]\}$ iff $\forall x(\varphi[x] = \psi[x])$
- Hence generally (i.e. naively): $\exists y \forall x(x \in y$ iff $\varphi[x])$
- Hence: $y \in y$ iff $y \notin y$

. . . the Fregean as well as post-Fregean meta-mathematical efforts are generally accepted as unifying the concept of analyticity.

Classical Foundations of Logic

Logical vs. Non-Logical: The Problem

We allow for a variable interpretation of the non-logical vocabulary, but for none (or only a fixed one) for the logical vocab.

E.g.: $\mathcal{I}(\varphi) \in \{0, 1\}$, $\mathcal{I}(F^n) \subseteq \mathcal{D}^n$, $\mathcal{I}(c) \in \mathcal{D}$

But, e.g.: Some kind of:
$\mathcal{I}(\&) = \{\langle 0, 0, 0\rangle, \langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 1, 1, 1\rangle\}, \mathcal{I}(\forall) = \dots$

Why not, e.g., $\mathcal{I}(\&) \in \{\{\langle 0, 0, x\rangle, \langle 1, 0, y\rangle, \langle 0, 1, z\rangle, \langle 1, 1, w\rangle\} : x, y, z, w \in \{0, 1\}\}$?

So we are looking for a justification of our demarcation between logical and non-logical vocabulary.

That's the question for providing an even more fundamental basis of reasoning.

Quine's Salva Congruitate Approach

According to Quine (1986), logical symbols are those symbols that are elements of the smallest categories of a language's expressions.

Such categories are formed by *salva congruitate* substitution. E.g. adverbs: ‘Willard philosophizes happily.’ $\Rightarrow$ ‘Willard philosophizes [happily/perfectly].’, but not, e.g.: ‘Willard philosophizes [happily/Pegasus].’

Logical categories are the smallest categories of the language under investigation.

Quine's Salva Congruitate Approach: Problems

Quine's theory has several problems:

- As an explication it is inadequate:
  - The set of n–ary sentence operators forms logical categories. ✓
  - The set of quantifiers forms logical categories. ✓
  - The set of individual constants and relation symbols forms non-logical categories. ✓
  - The set of individual variables forms non-logical categories (although perhaps proper in natural language). *X*
  - The set of atomic formulas forms non-logical categories.
    Problem cases: $\bot, \top$ *X*
  - The identity symbol is no logical symbol. *X*
- A more fundamental problem: The concept of a formula is presupposed in order to figure out *salva congruitate* substitutions.

Tarski's Invariance Approach

According to Tarsi (1986), logical symbols are those symbols that are most neutral with respect to the topic in question.

Background idea: Logic is the most fundamental scientific discipline and language we have. If you take all expressions of all sciences' languages and figure out which expressions appear in all of them, then you should get the most topic-neutral ones.

Technically seen, topic-neutrality is expressed as invariantly operating on permutations of the domain in question.

Just for the idea how this works:

- We label all elements of $\mathcal{D}$ with $d_1, d_2, \ldots$
- We perform our operations, e.g., $\mathcal{I}(F) = \{d_1\}, \mathcal{I}(c_1) = d_1$ and $\mathcal{I}(c_2) = d_2$; so $\mathcal{I}(Fc_1) = 1$, $\mathcal{I}(Fc_2) = 0$; also $\mathcal{I}(Fc_1 \& Fc_2) = 0$.
- By permuting our labelling of the elements of $\mathcal{D}$, of course our $\mathcal{I}$–operation on F is not invariant under such a permutation, whereas the &–operation is.

Tarski's Invariance Approach: Results and Problems

Tarski's theory has the following advantages:

- As an explication it is adequate. ✓
- It's a nice fleshing out of the topic-neutrality thought. ✓
- There is also a general theorem on the correctness of the theory: McGee (1996) showed that every operation on $\mathcal{D}$ that is invariant under permutation can be also defined by the standard logical operations (plus infinite disjunction). ✓

But it has also some problems:

- The invariance of logical operations does not hold under domain-size transformations. Put differently:
  Why should the "size" of the "universe" ($\mathcal{D}$) matter? X
- There are some strange logical operations as, e.g., H_2O–negation ($H_2O\varphi$ iff $\sim\varphi$&'Water is H_2O.')—cf. McGee (1996). X

For details cf. (Sher 2008).

Belnap's Structural Rules Approach

According to Belnap (1962), logical symbols are those symbols that can be introduced into a basic system by introduction- and elimination rules.

Background idea: If the way an operation works can be "explained" by help of basic logical rules alone, then it is argumentatively/logically relevant.

The basic logical system is just a system with the usual structural rules:

- Reflexivity: $\varphi \vdash \varphi$
- Weakening: If $\varphi_1, \ldots, \varphi_n \vdash \psi$, then $\varphi_1, \ldots, \varphi_n, \chi \vdash \psi$
- Contraction: If $\varphi_1, \varphi_1, \ldots, \varphi_n \vdash \psi$, then $\varphi_1, \ldots, \varphi_n \vdash \psi$
- Permutation & Transitivity

Then the connectives and quantifiers may be introduced by help of their usual introduction- and elimination rules. E.g.:

$\&I$: $\varphi, \psi \vdash \varphi \& \psi$ $\qquad$ $\&E$: $\varphi \& \psi \vdash \varphi$, $\varphi \& \psi \vdash \psi$

Belnap's Structural Rules Approach: Open questions

A problem and a solution of Belnap's theory:

- Problem:
  - *Tonk*-operator $*$ (cf. Prior 1960): $*I$: $\varphi \vdash \varphi * \psi$; $*E$: $\varphi * \psi \vdash \psi$.
  - So we get $\varphi \vdash \psi$ which is absurd.
- Belnap's solution:
  - Only conservative rule-extensions are allowed.
  - Where 'conservative' means that every derived (after adding the new operator) rule that contains no new operator, can be already derived by means of the structural (and beforehand added) rules alone.
  - Clearly adding the *tonk*-rules is no conservative extension since $\varphi \vdash \psi$ cannot be proven by means of the structural rules alone.

Open questions:

- There is no result on the general correctness of this approach. *X*
- There is a need of further justification of the structural rules. *X*

We will go on now with an approach concerned with the latter question.

Conventional Foundation of Logic

Theory of Definitions: Criteria

There are two classical constraints for conventions:
The complete "reduction" of usages by the criterion of eliminability:

Definition (Eliminability)

s is eliminable in T' w.r.t T iff for all $\varphi \in \mathcal{L}_{T,s}$ there is a $\psi \in \mathcal{L}_T$ such that: $\vdash^{T'} (\varphi \leftrightarrow \psi)$.

So, for every $\mathcal{L}_{T,s}$-claim there must be a $\mathcal{L}_T$-claim that is T'-equivalent in order to satisfy eliminability of s.
The constraint of not smuggling in knowledge by the criterion of non-creativity/conservativity (cf. Belnap's solution before):

Definition (Non-creativity)

T' is a non-creative extension of T iff for all $\varphi \in \mathcal{L}_T$ it holds: $\vdash^{T'} \varphi$ iff $\vdash^{T} \varphi$.

So, old usages (extensions) remain unchanged.

A Partial Definitional Basis of Logic

So, recall, we are looking for a further foundation of the structural rules of the rules-approach to logical symbols.

We will do so by providing a set of “definitions” that should suffice to “imitate” these rules (and allows also for a definition of the connectives).

A speciality: We define inference-relations ($\mathcal{R}$) since here the definitional rules are well investigated and they are pre-theoretically harmless:

- Partial definition of $\mathcal{R}^1$: $x = y \Rightarrow x\mathcal{R}^1 y$
- Circular addition for $\mathcal{R}^1$ (transitivity): $x\mathcal{R}^1 y$ and $y\mathcal{R}^1 z \Rightarrow x\mathcal{R}^1 z$
- Characterisation of $\mathcal{R}^2$:
  - Part. def. of $\mathcal{R}^2$ (contraction): $x_1 = x_2 \Rightarrow x_1, x_2\mathcal{R}^2 y$ iff $x_1\mathcal{R}^1 y$
  - Part. def. of $\mathcal{R}^2$ (weakening): $x_1\mathcal{R}^1 y \Rightarrow x_1, x_2\mathcal{R}^2 y$
  - Circular addition for $\mathcal{R}^2$ (right-transitivity): $y_1\mathcal{R}^1 y_2 \Rightarrow (x_1, x_2\mathcal{R}^2 y_1 \Rightarrow x_1, x_2\mathcal{R}^2 y_2)$
  - Circular addition for $\mathcal{R}^2$ (left-transitivity and permutability):
    $x_1\mathcal{R}^1 x_2 \Rightarrow (x_2, x_3\mathcal{R}^2 y \Rightarrow x_3, x_1\mathcal{R}^2 y)$
- Definition of $\mathcal{R}^n$ ($n \geq 3$) later on with &:
  $x_1, \ldots, x_n\mathcal{R}^n y$ iff $x_1, x_2\& \ldots \& x_n\mathcal{R}^2 y$

A Partial Definitional Basis of Logic

With the help of these inference relations, one can define the connectives also explicitly. E.g.: $x \& y =_{\mathcal{R}} z$ iff $z \mathcal{R}^1 x$ and $z \mathcal{R}^1 y$ and $x, y \mathcal{R}^2 z$

NB: Of course one cannot prove for all x, y that there is a unique z satisfying the definiens. But one can prove that all such z_1 and z_2 satisfying the definiens are $\mathcal{R}$–equivalent, i.e.: $z_1 \mathcal{R}^1 z_2$ and $z_2 \mathcal{R}^1 z_1$.

NB too: The multiple partial characterisation of $\mathcal{R}^2$ is still non-creative since their combination is already logically valid.

Theorem (Observation)

- *The definitional characterisation is correct.*
  (Replacing the $\mathcal{R}$s by $\vdash$ results in correct rules.)
- *It is also complete (w.r.t. propositional logic pl).*
  (Due to compactness of pl we can re-write every pl-proof in our $\mathcal{R}$-notation; for all $\mathcal{R}$s there hold the respective structural rules (weakening, etc.); intro.- and elimination rules as in the definitions.)

Summary

1. Starting point: *analytical* = *mathematical* + *logical* + *conventional*
2. The logicistic programme: *mathematical* ⇒ *logical*
3. In case of success: *analytical* = *logical* + *conventional*
4. There is a problem of justifying *logical* with three approaches resulting:
    - Substitution *salva congruitate*
    - Invariance under domain-operation permutations
    - Conservative expansion of structural rules
5. We sketched how one may try to (partially): *logical* ⇒ *conventional*
6. By this one my overcome the justification-problem of the structural rules approach
7. And one may end up with: *analytical* = *conventional*

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